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SEMIFREE LOCALLY LINEAR PL ACTIONS ON THE SPHERE

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ABSTRACT
This paper aims at giving a complete classification of semifree orientation preserving PL locally linear group actions on the sphere.

The object of this paper is to give a complete classification of semifree orientation preserving[††] PL locally linear group actions on the sphere. We succeed in case the fixed set is assumed to be of codimension larger than two.[‡] Our main results are the following:

THEOREM A. *A PL locally flat submanifold Σ^n of S^{n+k} for $k > 2$ is the fixed set of an orientation preserving semifree PL locally linear G action on S^{n+k} iff Σ is a $\mathbf{Z}/|G|$ homology sphere, $\mathbf{R}^k$ has a free linear representation of G, and certain purely algebraically describable conditions hold for the torsion in the homology of Σ.*

THEOREM B. *Two orientation preserving semifree PL locally linear G actions on S^{n+k} with Σ as fixed set differ by equivariant connected sum with a semilinear sphere[‡‡] iff the equivariant Atiyah–Singer classes for the two actions coincide.*

To make these theorems more explicit one needs some definitions. We denote by $\tau(\Sigma)$ the product $\Pi|H_{2i}(\Sigma\text{-point})|/\Pi|H_{2i+1}(\Sigma\text{-point})|$. This is a

[†] Partially supported by an NSF PYI award and a Sloan Foundation Fellowship.
[††] For groups other than $\mathbf{Z}_2$ semifree implies orientation preserving.
[‡] This is a restriction only for cyclic groups.
[‡‡] A semilinear sphere is a sphere with group action with the property that the fixed set of each subgroup is a sphere.

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multiplicitive analogue of the usual Euler characteristic. Poincare duality implies that it vanishes (i.e. *equals one*) if n is even. In this case, one defines $\tau_{1/2}(\Sigma)$ to be the same product, but with indices only running through integers at most half the dimension of the manifold. In Theorem A we assert that the question of whether or not Σ is a fixed set can be determined purely algebraically from τ and $\tau_{1/2}$. (For odd order groups the condition is that the Swan homomorphism applied to τ is trivial; for even order groups see Section 2.) Notice that the dimension of the ambient sphere or the exact embedding of Σ in it are both irrelevant. This is certainly not the case if $k = 2$.

The Atiyah–Singer classes referred to in Theorem B were constructed in [CW1] and enter in a PL G-signature theorem. They are elements of the cohomology theory determined by the localized L-spectrum, $\tilde{\mathbf{L}}^s(\mathbf{Z}G)[1/|G|]$, of grading k. We also determine which elements of the cohomology theory arise as characteristic classes for some action. (This is a four periodic theory, and the answer is also four periodic (assuming that there is a four dimensional free representation of G).)

If one studies topological locally linear actions rather than PL actions, then the analogue of Theorem A has been known for three years already [We1]; Σ is a fixed set iff it is a $\mathbf{Z}/|G|$ homology sphere and $\mathbf{R}^k$ has a free linear representation of G. (The classification of topological actions with given fixed sets is more subtle and will be the subject of another paper; see [We3].) I do not know if it is possible to construct a proof of Theorem A using this and equivariant triangulation theory [LR].

On the other hand, the smooth case seems very difficult and involves a wide variety of different insights, ranging from improvements on the present technique to analyses of unstable homotopy theory of compact Lie groups and stable homotopy theory, to be able to attack it perpicaciously.

These theorems have a long history. Here are some of the highlights. In his famous 1944 paper, P. A. Smith [Sm] showed the necessity of the homological condition. (The condition on the representation is part of the definition of local linearity.) The whole possibility of obtaining positive existence and classification results seemed unlikely until L. Jones [J] made a major breakthrough and gave examples to show that only $\mathbf{Z}/|G|$ homology can be controlled. He classified smooth† fixed sets in high codimension for cyclic

† The smooth case is very different, because of the theorem of Connor and Floyd that asserts the existence of a complex structure on the normal bundle to the fixed set of any smooth semifree action of any group other than $\mathbf{Z}_2$ on any smooth manifold.

group actions on the disk, and made some progress on the sphere. A. Assadi showed that a condition on $\tau(\Sigma)$ is necessary for PL (and, *a fortiori*, smooth) fixed sets for the more general groups. In 1978, R. Schultz [Sc] (building on work of Jones, Alexander, Hamrick, and Vick [AHV]) completed the problem in the smooth case of cyclic actions, using then recent work on the Segal conjecture. (P. Loffler [L] had earlier related the Segal conjecture to the special case of which *homotopy spheres* can be fixed sets.) The case of smooth actions on the disk was then solved in general† in unpublished work by Assadi and W. Browder (not necessarily assuming high codimension) and, independently, slightly later by the author [We2]. The PL locally linear case was completed jointly with S. Cappell afterward and is a key ingredient in the present proof. At the same time we proved Theorem A for G cyclic of prime power order using [CW2]. Soon after I noticed that the improvement of [CW2] due to J. Davis and Loffler [DL] could be used to handle all cyclic groups and all groups of odd order. I also realized then that one could do the topological locally linear case by the same method that Cappell and I used for the PL locally linear case on the disk with a new wrinkle. In the topological category one can do infinite constructions to avoid obstructions like Assadi's condition on τ and other surgery obstructions which we couldn't compute. (This was inspired by F. Quinn's examples [Q] of topological locally linear actions of non-p-groups on the disk that had nontrivial equivariant finiteness obstructions and with homotopy types of fixed sets that could not arise for PL actions because of work of R. Oliver [O].) The present theorem is the first case that requires control of $\tau_{1/2}$ and was done in late 1985 making use of pieces of several earlier aproaches as well as Davis' theory of numerical surgery invariants [D].

Our proof (and paper) is organized as follows. First we use [CW1] to obtain an action on D^{n+k} with a punctured copy of Σ as fixed set. We then try to analyze when the action on the boundary sphere can be taken linear. Ultimately two interpretations of the obstruction are necessary: one as a torsion, and one as a surgery obstruction. The torsion approach is sufficient to handle groups of odd order. Then following the smooth idea in [Sc] (see also [CW2])

† Modulo difficult lifting problems in classical homotopy theory of the type necessitated in the previous footnote. One case though where one gets a clean answer is Z_2. A smooth submanifold (properly embedded) in the disk of codimension other than two is the fixed set of an orientation preserving smooth involution iff it is Z_2-acyclic. Under those hypotheses the involution is unique up to conjugacy.

we reduce the problem to a bordism problem, again using [CW1]. For this to be feasible it is first necessary to show that the problem does not depend on the ambient dimension, and consequently on the exact embedding of Σ. It is also important for reasons of computability that we do not have to control the equivariant normal bundle in the bordism theory as was necessary in [Sc] (as much less is known about BPL than BU). Since G is now assumed to have even order, all homology spheres are mod 2 homology spheres and the 2-torsion in the bordism theory can be directly computed. This enables us to reduce to the case where the fixed set is stably parallelizable so that the propagation ideas of [CW2] together with the improved method of computing surgery obstructions [D] can be applied to compute the role of the obstruction in the bordism theory. Theorem B follows from the analogous result for the disk in [CW1]. I remark that the classification of semilinear PL locally linear spheres was done by Rothenberg and Sondow in the 1960's and will be summarised (*and applied*) below.

I would like to thank all the many mathematicians whose work I have applied or built upon and those who have encouraged me over the past several years during different stages of this program. I especially thank Sylvain Cappell and Jim Davis, without whom this project would still seem (to me) impossible. Finally, it is appropriate to dedicate this paper to Alex Zabrodsky, a friend as well as colleague, whose beautiful ideas are fundamental for what follows.

§1. Odd order groups

In this section we shall prove Theorems A and B in case G is a group of odd order. In partial outline the proof is the same as the even order case. The difference is the endgame. In the odd order case we give a direct calculation of the final obstruction (which is identically zero in this case), while in the even order case it is necessary to work rather indirectly through a reduction to a cobordism calculation. It is interesting to note that the even order proof does not apply to the odd order case since the relevant cobordism group then has much too much 2-torsion (reflecting the 2-torsion present in the homology of fixed sets of odd order group actions on the sphere).

THEOREM 1. *Let G be an odd order group; a PL locally flat submanifold Σ^n of S^{n+k} for $k > 2$ is the fixed set of an orientation preserving semifree PL locally linear G action on S^{n+k} iff Σ is a $\mathbf{Z}/|G|$ homology sphere, $\mathbf{R}^k$ has a free linear representation of G, and $\sigma(\tau(\Sigma)) \in K_0(\mathbf{Z}G)$ is zero.*

Henceforth we shall denote $|G|$ by Γ.

The definition of τ was given in the introduction: it is an alternating product of the orders of the torsion in the homology of Σ. This is a fraction with numerator and denominator relatively prime to Γ, and thus defines an element of $(\mathbf{Z}/\Gamma)^*$. The homomorphism σ maps this group into K_0, and is usually called the Swan homomophism. One definition is that it is the boundary map in the Mayer–Vietoris sequence of the square:

$$\begin{array}{ccc} \mathbf{Z}G & \longrightarrow & \mathbf{Z}G/N \\ \downarrow & & \downarrow \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}/\Gamma \end{array}$$

where N is the norm element, i.e. the sum of all the elements of the group. Another definition, often more useful, but readily seen to be equivalent, is based on viewing K_0 as being defined by modules of finite projective length (rather than projective modules) and then the map from $(\mathbf{Z}/\Gamma)^*$ is defined by considering the element of K_0 determined by the module with trivial G action of the given order. (Such a module has finite projective length according to a theorem of Rim. The above square is often called the Rim square.)

The necessity of the vanishing of $\sigma(\tau(\Sigma))$ is given in [A]. The main object of this section is to prove the sufficiency statement of the theorem. The proof given here is a large simplification of the one sketched in [We1].

We begin with the classification of semilinear semifree PL spheres up to concordance.

LEMMA 1. *Semifree PL locally linear G actions on S^{n+k} with fixed set S^n, and given normal representation, up to concordance are in a one to one correspondence with $H^{n+k}(\mathbf{Z}_2; \mathrm{Wh}(\mathbf{Z}G))$. The class of linear representation is represented by* 0.

PROOF. This is a weakening of the result of Rothenberg and Sondow [RS] proven by the same method (twenty years later)! Since we are in the PL category the operation of removing an open disk is reversible by coning the boundary. Actions on the disk with fixed set a disk and given normal representation are in a one to one correspondence with $\mathrm{Wh}(G)$ by taking the torsion of the inclusion of a normal fiber in the complement, by an application of the h-cobordism theorem. The fact that the boundary is linear forces the torsion to be self-dual. Replacing an action by a concordant one changes this torsion by the norm of the cobordism. □

We shall apply this to the following situation: Puncture Σ and use [CW1] to get this as a fixed set on the disk. (This already requires that Assadi's condition $\sigma(\tau(\Sigma)) = 0$ holds.) Then examine the action on the boundary sphere and show that it is linear. (For G of even order it won't always be.) We call the above invariant of a semilinear sphere S, $T(S)$.

Let us recall Bass's exact sequence of a localization for lower K-theory ([B]):

$$K_1(\mathbf{Z}G) \to K_1(\mathbf{Z}_{(\Gamma)}G) \to K_1(\mathbf{Z}G, \mathbf{Z} - \Gamma) \xrightarrow{\sigma} K_0(\mathbf{Z}G) \to K_0(\mathbf{Z}_{(\Gamma)}G)$$

where the middle term is the Groethendieck group finite projective length $\mathbf{Z}G$ modules that are annihilated by some integer relatively prime to Γ. For example, we can view the homology modules of $\hat{\Sigma}$, that is, Σ punctured, as elements of this group. We shall in particular be interested in $\tau(\Sigma)$ in $K_1(\mathbf{Z}G, \mathbf{Z} - \Gamma)$. (Notice that although the definition of σ does not agree with the one given above, $\sigma(\tau)$ is independent of the disagreement.)

The way that τ enters is via the work of [AB] (see also [AV]), [We1] and [CW] on extensions across homology collars. The relevant theorem for us is from the final reference and deals with the problem of extending certain semifree group actions on manifolds with boundary.

The setting is this. A manifold with boundary $(W, M \cup N)$ is a Λ-homology collar if $H_*(W, M; \Lambda) = 0$. We assume for simplicity that W and N are simply connected. (If not, then one can jazz things up slightly using [AV].) Suppose that G acts semifreely, PL locally linearly, and $\mathbf{Z}[1/\Gamma]$-homologically trivially on M, and W is a $\mathbf{Z}_\Gamma$-homology collar. One can extend the action to one on W (with a desired fixed set) iff certain cohomological conditions related to Atiyah–Singer classes, and certain K-theoretic conditions involving $\sigma(\tau)$, hold.

THEOREM (Hard Extension Across Homology Collars [CW]). *Under the assumptions made in the previous paragraph, an action with fixed set K, not of codimension two, is extendible to such a homologically trivial action on W with fixed set L iff*:

(1) $H_*(L, K; \mathbf{Z}_\Gamma) = 0$,

(2) $\sigma(\tau(W, M)/\tau(L, K)) = 0$,

(3) *the Atiyah–Singer classes associated with the action on M, lying in $[K : \tilde{\mathbf{L}}(G)[1/\Gamma]]$, extend to classes lying in $[L : \mathbf{L}(G)[1/\Gamma]]$.*

Furthermore, concordance classes of extensions are classified by the extensions of the Atiyah–Singer data.

L denotes the Quinn surgery space for simple surgery (of dimension =

codimension of the fixed set). Suitable relative versions are easily formulated, and are true. To obtain a submanifold of the disk as a fixed set one views the disk as a homology collar with M = a linear D^{n-1} on the boundary. We consider the result of this construction on Σ', a punctured copy of the $\mathbf{Z}_\Gamma$-homology sphere, and consider the semilinear sphere on the boundary.

We need two more maps:

$$\delta : H^*(\mathbf{Z}_2; K_0(\mathbf{Z}G)) \to H^{*+1}(\mathbf{Z}_2; \mathrm{Wh}(G))$$

and

$$\varepsilon : H^*(\mathbf{Z}_2; \ker \sigma) \to H^{*+1}(\mathbf{Z}_2; \mathrm{Wh}(G)).$$

To define the first, one uses the fact that the norm of any projective class in the cohomology is 0, so that one can write down an obvious hyperbolic form, whose discriminant is the desired element of the target. The second is just the map given by the snake lemma applied to Bass's localisation sequence.

LEMMA 2. $T(S) = \pm\, \delta[\sigma(\tau(\Sigma))] = \pm\, \varepsilon[\tau(\Sigma)]$.

Clearly (either expression for $T(S)$ given in) the proposition implies the classification Theorem B for odd order groups in light of its independence of the equivariant neighborhood of Σ chosen. Theorem 1 also follows, for observe that in odd dimensions the cohomology group vanishes, while in even dimensions Poincaré duality implies that $\tau \equiv 1$. We shall only prove the second expression for $T(S)$ although the first should be suggestive of a more direct surgery theoretic approach, which we will pursue in the next section on even order groups. (The equality of the two expressions can be taken as an algebraic exercise whose solution we, in any case, have no use for.)

The proof of this equality is an exercise in the duality theorems of Poincaré-Lefshetz and for torsions, and the formula for torsions of unions. For completeness, here is the calculation:

$$T(S) = \tau(S^{c-1}/G \to (S^{n+c-1} - S^{n-1})/G) \qquad \text{(definition)}$$

$$= N\tau(\text{(boundary of a neighborhood of } \Sigma') \to \text{complement}) \qquad \text{(duality)}$$

so a preimage of $T(S)$ under ε is the final expression. (Notice that we have pushed into $K_1(\mathbf{Z}_\Gamma G)$ by now, which for even order groups can be quite a problem even as far as cohomological calculations are concerned.) The last can be computed using the fact that Euler characteristics can be computed using homology as well as chain complexes:

$= N\tau(H_*(D - \Sigma', \text{(boundary of a neighborhood of } \Sigma')))$

$= N\tau(\Sigma)^{\pm 1}$. (duality)

The result now follows.

REMARKS. One should compare with [DL] (see the introduction for historical remarks). Also, with little extra work one sees that Theorem 1 is actually correct for G with normal cyclic two sylow subgroup.

§2. Even order groups

Our first object is to make the statement of Theorem A more precise. As before, Theorem B is proven along the way to proving Theorem A.

THEOREM A′. *Let $\rho : G \to \mathrm{SO}(k)$, be an orientation preserving semifree representation of a finite group G of even order. Let $\Sigma^n \subset S^{n+k}$ be a properly embedded PL submanifold. There is a semifree PL locally linear G action on S^{n+k} with fixed set F iff*

(i) $H_*(\Sigma'; \mathbf{Z}_{(G)}) = 0$,

(ii) $\sigma(\tau(\Sigma)) = 0 \in K_0(\mathbf{Z}G)$,

(iii) *the image of the numerical surgery element of $L^h(\mathbf{Z}G)$ associated with $H_*(\Sigma)$ is zero in $H^*(\mathbf{Z}_2; \mathrm{Wh}(\mathbf{Z}G))$.*

The numerical surgery element is defined depending on the dimension as follows: If Σ is even dimensional, say dimension $2k$, one defines $\tau_{1/2}(\Sigma)$ to be the alternating product of the order of the homology of Σ from dimension one through $k - 1$. (This should be viewed as analogous to what one does with Betti numbers: the Euler characteristic is valuable only in even dimensions, because duality implies vanishing in odd dimensions, so in odd dimensions one introduces the semicharacteristic which is the alternating sum of half of the homology.) One then applies the Swan homomorphism to this element to get an element of K_0 which defines an element in cohomology, and thus in L-theory via the Rothenberg sequence. In dimension $4k + 1$ one observes that $H_{2k}(\Sigma)$, in virtue of its possession of a skew-symmetric linking form, is a square (since it's odd order), consequently $\tau(\Sigma)$ is, and one does the same thing to its† square root. Finally, in dimension $4k + 3$, following Davis [D], one must be

† In [We1] it is erroneously stated that one computes the square root of H_{2k}. (See the disclaimer on the top paragraph of p. 271 of that paper, please.)

more devious. Davis first shows that in dimension 3, $L^h(\mathbf{Z}G) \to L^A(\mathbf{Z}_\Gamma G)$ injects where L^A denotes an intermediate Wall group and A is the image of $K_1(\mathbf{Z}G)$ (= ker σ). Thus to determine an element of $L^h(\mathbf{Z}G)$ I must specify an element of $L^A(\mathbf{Z}_\Gamma G)$, and this is determined by viewing $\tau(\Sigma)$ as an element of $K_1(\mathbf{Z}G, S)$, and as usual by now, pushing forward in an appropriate Rothenberg sequence. (This element only depends on τ mod 4Γ.) These elements are defined so that if one has certain sorts of surgery problems with surgery kernels finite of order prime to Γ, then the surgery obstruction is the given element. The reader should compare [D, We, DW1, CW] for more information and applications.

We now sketch the argument.

Step I (Independence of Atiyah–Singer data). Let O denote $T(S)$ for a given choice of local data around a punctured fixed set. Suppose we have two different choices of Atiyah–Singer data, α and β, one proves that the boundary sphere has all the same simple homotopy data for these two choices, so that $O(\Sigma, \alpha) = O(\Sigma, \beta)$. We are now justified in referring to the obstruction as $O(\Sigma)$.

REMARK. As in the previous section, Theorem B follows at this point from hard extension.

Step II. $O(\Sigma_1 \# \Sigma_2) = O(\Sigma_1) + O(\Sigma_2)$. This is just a picture and the addition formula for Whitehead torsion.

Step III (Stability). One computes (using the product and sum formulae for torsion) to see the obstruction does not change if one increases the dimension of the sphere.

Step IV (Cobordism invariance). Let us examine homology spheres satisfying conditions (i) and (ii) up to cobordisms satisfying the same conditions. One makes use of the invariance under change of Atiyah–Singer data and hard extension to handle the obstruction.

Step V. Now, one invokes the computation [AHV] of the cobordism of PL $\mathbf{Z}_{(\Gamma)}$ homology spheres to produce, at the prime 2, stably parallelisable generators. These one handles by finding a transverse isovariant map of pairs $(S^{n+k}, F) \to (S^{n+k}, S^n)$ (raising k if necessary (step III)). At this point one invokes the theory of homology propagation to decide when one can use this map to construct the necessary actions. Theorem A follows.

Now for the necessary details.

LEMMA 3. *The equivariant homotopy type of the action on normal bundle to Σ is independent of the choice of Atiyah–Singer data.*

PROOF. At Γ the normal fibration is fiber homotopically trivial since the action extends to the complement, which is a splitting by Alexander duality. Away from Γ the homotopy theory, by nilpotence, is detected by the unequivariant theory, which is pinned down by the embedding.

As a consequence, the obvious induction using local linearity shows that the boundaries of the regular neighborhoods are simple homotopy equivalent. Step I now follows by Milnor duality — the norm of the torsion of the homotopy equivalence between the extensions to the complements is the difference between the linear obstructions of the boundary spheres, so they are concordant.

Steps II and III are entirely routine and can certainly be left to the reader.

Step IV unfortunately must be handled by a case-by-case analysis. We need a homological lemma analogous to well known properties of Euler characteristics and semicharacteristics.

LEMMA 4. *Let Δ be a $\mathbf{Z}_{(2)}$-homology disc of dimension d and Σ its boundary. Then:*

(i) *if d is* 1 mod 4 *then $\tau(\Delta)/\tau_{1/2}(\Sigma)$ is a rational square,*

(ii) *if d is even then $\tau(\Delta)^2 = \tau(\Sigma)$.*

Note that one can apply the lemma to homology spheres by puncturing them. We leave the lemma's proof to the reader. (See [K] for the analogue of (i) for the Euler characteristic.)

PROPOSITION 1. *Suppose one has a homology collar situation where the Atiyah–Singer classes extend but the Swan condition does not hold. Then one can construct an action on the other end compatibly with the remaining data iff the image of $[\sigma(\tau(W, M)/\tau(L, K))]$ in $L^h(\mathbf{Z}G)$ vanishes.*

PROOF. If the element actually vanished, then one could extend across the homology collar. However if it does not, then the proof of extension across homology collars (see [We]) produces a projective coboundary of a surgery problem on the other end whose ordinary solution would produce the action. The finiteness obstruction of the projective coboundary is easily identified with this element, so the result follows from the definition of the map in the Rothenberg sequence.

Cobordism invariance, except in dimension $4k + 2$, now follows from the lemma and proposition since, by using step I, we can assume that all the Atiyah–Singer data is trivial. In dimension $4k + 2$ one stabilises first and then connect sums the cobordism with an appropriate lens space to obtain the Swan condition.

At this point we actually have that for a Swan homology sphere the difference between $O(\Sigma)$ and the element described in the theorem is an invariant of bordism of $\mathbf{Z}_\Gamma$ homology spheres.

To complete the final step observe that we therefore have a homomorphism (in light of step II) from this bordism (or at least the subgroup represented by Swan spheres) to $H(\mathbf{Z}_2; \mathrm{Wh}(\mathbf{Z}G))$, a 2-group. We localise at 2. Then the bordism group is given by an analogue of the usual surgery exact sequence and one gets

$$L_{n+1}(\mathbf{Z}_\Gamma) \to \Omega_n^{HS}(\mathbf{Z}_\Gamma) \to 0$$

(see e.g. [AHV] for why the next term is infinitely generated odd torsion). Therefore, modulo an easily eliminated Kervaire obstruction, one obtains the result unless n is 3 mod 4. In that case the L-group is infinitely generated. Therefore, the only homology spheres that we have to examine are produced by a Wall realisation to the standard sphere, and are in particular stably parallelisable.

Raising dimensions to be in a stable range (step III) Spanier–Whitehead theory produces a map $(S^{n+N}, \Sigma) \to (S^{n+N}, S^n)$. View this as a transverse isovariant propagation problem as in the final section of [CW1]. Using the result of Davis cited above we can do the calculation of the necessary surgery obstruction in $L^A(\mathbf{Z}_\Gamma G)$. The argument now is similar to one in [DW2] (see also the papers [CW3, DW1, We1] for cases where there are no boundary contributions).

The general machinery produces a degree one normal invariant for the homotopy exterior (constructed by Zabrodsky mixing) which is a homeomorphism on the boundary. The surgery obstruction of this normal invariant, pushed into $H^*(\mathbf{Z}_2; \mathrm{Wh}(\mathbf{Z}G))$, is $O(\Sigma)$ by an analogue of Proposition 1, proven exactly the same way. We must first relate this normal invariant to a normal invariant for the linear action by making use of a normal cobordism from Σ to S. The following is very useful:

PROPOSITION 2 ([D2]). *If one takes the product of any simply connected local normal invariant with a space form, the obstruction in $L_3^A(\mathbf{Z}_\Gamma G)$ lies in the image of the assembly map.*

Davis deduces this from [DW1] by showing that that paper computes the result of surgering products of space forms with nondegree one maps between zero manifolds, and then using these maps to generate the simply connected local L-group.

We can, using this proposition, glue onto both domain and range the normal cobordism from Σ to S crossed with S^{k-1}/G without changing any surgery obstructions, and the composite map to the linear model lies in the image of ooze (see [CW3, DW1, DW2, We1]) since the map is a homeomorphism on the boundary. Modulo ooze, then, the obstruction for the original surgery problem is the negative of the surgery of the complement to the linear complement which is, by the numerical theory of [D1], its τ, which is, by duality, up to sign, $\tau(\Sigma)$. Since the image of ooze vanishes in $H^*(\mathbf{Z}_2; \mathrm{Wh}(\mathbf{Z}G))$ (since manifolds have canonical structures of simple Poincaré complexes) the ambiguity does not effect the image of the surgery obstruction in the cohomology. The proof is complete.

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